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The exact solution of a non-linear boundary-value problem of the theory of waves on the surface of a liquid of finite depth☆

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ABSTRACT

A non-linear boundary-value problem of the theory of waves on the surface of a heavy ideal incompressible liquid, which arises as a result of the expansion of the required functions in amplitude, taking quadratic terms into account, is investigated. A solution is constructed, on the one hand, suitable for describing long waves, and on the other, matched to the Stokes expansion (i.e., with the expansion in amplitude of the first order of infinitesimals). A function is sought which conformally maps a strip into the plane of the complex potential in the flow region. An exact solution is obtained for this problem, defined by fairly simply formulae. This solution, in the limit of long and short waves, gives linear sinusoidal waves and cnoidal waves respectively.

In the theory of stationary gravitational waves of small amplitude, the Stokes expansion is well known and gives sinusoidal waves, and also the long-wave expansion, which gives cnoidal waves. Neither of these expansions is suitable for all wavelengths. For example, the Stokes expansion, in the limit of long waves, does not give a solitary wave, since, to obtain the Stokes expansion, amplitude-linearization is used, whereas solitary waves are essentially non-linear phenomena. When quadratic terms are retained, a quadratially non-linear boundary-value problem is obtained, which is analysed below. Its exact solution is obtained and, as a result, for small amplitudes an approximate solution is constructed that is equally suitable for all wavelengths.

1. The family of gravitational waves

We will consider the problem of waves on the surface of an ideal incompressible liquid of finite depth. The flow is plane parallel, potential and steady, and there is no surface tension. The liquid is bounded below by a rough bottom, and above by a periodic fixed free surface. We will seek a two-parameter family of solutions having one trough and crest per period. If we choose the amplitude and velocity of the wave as the free parameters, the range of permissible values of these parameters is a curvilinear triangle (for example, triangle *ABC* in Fig. 1). A certain gravitational wave corresponds to each point.

The range of small amplitudes has been most investigated. Here both the Stokes expansion is applicable which, in the first order, gives the theory of linear sinusoidal waves, and also expansion within the framework of shallow-water theory (long-wave theory). The problem becomes more complicated as the amplitude increases. The boundary *AB* corresponds to waves of maximum amplitude, having an angle of 120° at the crest. The theory of waves having maximum or close to maximum amplitude was obtained earlier in Refs 1–3. The boundary *BC* corresponds to solitary waves, i.e., gravitational waves of finite length. The point *B* is a singular point. It corresponds to a solitary wave of maximum amplitude, having an angle of 120° at the crest on the free surface. The point *C* is also a singular point, and in its neighbourhood the expansion within the framework of shallow-water theory holds, which, in the first order, gives cnoidal waves. The area of applicability of the Stokes expansion is concentrated in a narrow band along the horizontal axis. The point *A* is also a singular point – it corresponds to gravitational waves in infinitely deep liquid.

The Stokes expansion and the expansion within the framework of shallow-water theory arise from the assumption that two small parameters are present:

$$\epsilon \sim \frac{\text{wave amplitude}}{\text{liquid depth}}, \quad \varepsilon \sim \frac{\text{liquid depth}}{\text{wave length}}$$

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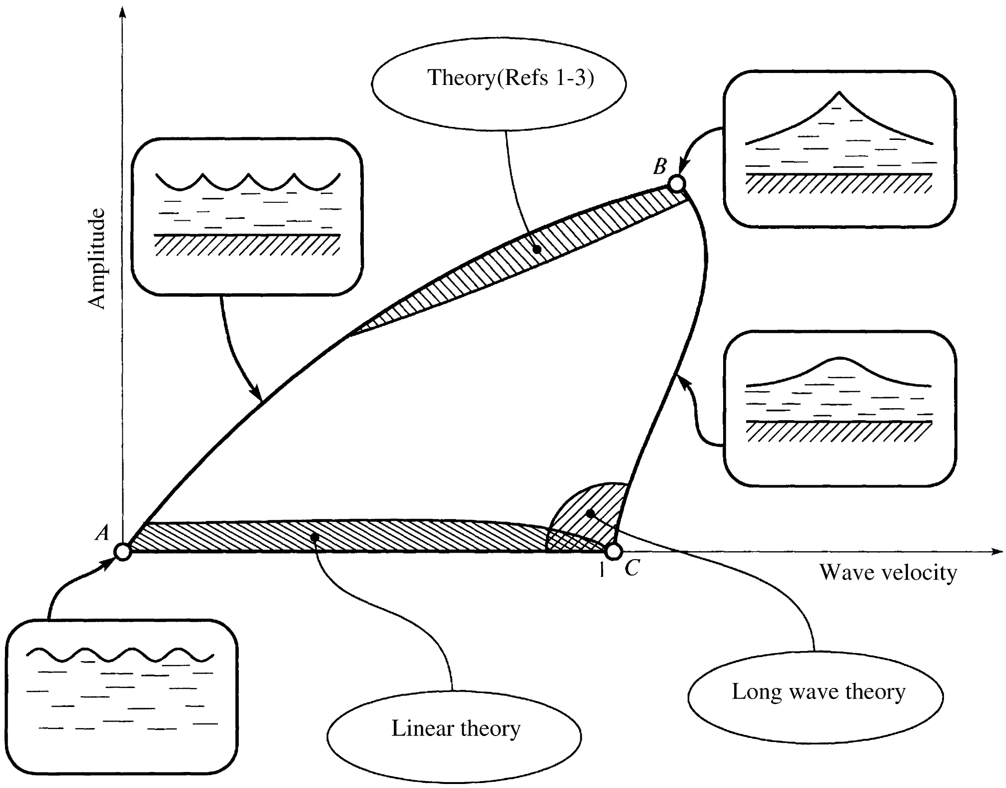

Fig. 1.

The construction of a solution of the wave problem can be simplified if we seek a function $f(\chi)$ which conformally maps the strip into the plane of the complex potential χ in the region occupied by the liquid. In fact, in this case, instead of a boundary-value problem in the region with unknown free boundary we must solve a simpler boundary-value problem in a fixed region – the strip.

If the function $f(\chi)$ is sought in the form of a power series in $\in$, we obtain the Stokes expansion

$$f = f_0(\chi) + \epsilon f_1(\chi) + \epsilon^2 f_2(\chi) + \epsilon^3 f_3(\chi) + \dots \tag{1.1}$$

If we first make an extension, i.e., change from the variable χ to the variable $\varepsilon\chi$, and then seek a solution in the form of a power series in $\in$, we obtain an expansion within the framework of shallow-water theory

$$f = \varepsilon^{-1} g_0(\varepsilon\chi) + \varepsilon g_1(\varepsilon\chi) + \varepsilon^3 g_2(\varepsilon\chi) + \varepsilon^5 g_3(\varepsilon\chi) + \dots \tag{1.2}$$

The case $\in = 0$ and $\varepsilon = 0$ corresponds to waves of zero amplitude. Hence, taking a finite number of terms of expansions (1.1) and (1.2) for small $\in$ and ε, we obtain approximate solutions for small amplitudes. However, these solutions do not hold for all wavelengths. Expansion (1.2) only describes long waves. In the short-wave limitm, although it also gives sinusoidal waves, the dispersion equation obtained (the relation between the velocity of the wave and its wavelength) is inaccurate. The Stokes expansion (1.1), on the other hand, does not describe long waves. For example, there is no limit transition to long waves. The question arises: is it not possible to construct an approximate solution that is suitable for all wavelengths? This paper is devoted to this question.

In expansion (1.1) the function $f_0(\chi)$ is found trivially, and all the remaining $f_j(\chi)$ are found from the solution of linear boundary-value problems. In this paper we will confine ourselves to finding $f_1(\chi)$.

In expansion (1.2) the function $g_0(\varepsilon\chi)$ is found trivially, the function $g_1(\varepsilon\chi)$ is found from the solution of a non-linear differential equation – it is obtained in this paper, while all the remaining $g_j(\varepsilon\chi)$ are found from the solution of linear inhomogeneous equations.

The approach used in this paper, is in fact based on an expansion in the amplitudes (1.1). It is proposed to retain all terms up to $O(\in^2)$ inclusive, which enables us to eliminate the non-linearity in the boundary-value problem. As a result, the modified Stokes expansion will describe such a non-linear phenomenon as a solitary wave.

The main result of this paper is the fact that this non-linear boundary-value problem can be solved exactly. The method of solution, proposed previously in Refs 1 and 2, turned out to be successful here.

2. Reduction of the boundary-value problem to a system of equations

For many non-linear wave problems, the periodic solution, which depends on the real variable φ, in the linear approximation is expressed in terms of the functions $\sin\varphi$ and $\cos\varphi$, which, in the slightly non-linear approximation (small amplitude), are replaced by elliptic functions, for example, the Jacobi functions $\mathrm{sn}\varphi$ and $\mathrm{cn}\varphi$. For a large amplitude, series in $\mathrm{sn}\varphi$ and $\mathrm{cn}\varphi$ appear. We can cite as an example non-linear wave trains or waves in a liquid of finite depth, considered in this paper. It is important that the elliptic functions for a complex value of the argument should be double periodic (the function $\mathrm{sn}(\varphi + i\psi)$ is periodic in both φ and in ψ). Therefore, in wave problems, in addition to the solution periodic in φ, we can also try to seek a solution periodic in ψ. Sometimes a search for a solution possessing an imaginary period is simpler than the search for a solution with a real period.

The general scheme of the discussion becomes more understandable if we consider the following boundary-value problem.

Problem 1. It is required to obtain a function $f(\chi)$ of the complex variable $\chi=\varphi+i\psi$, analytic in a strip of unit width

$$0<\psi<1, \quad -\infty<\varphi<\infty \tag{2.1}$$

and which satisfies the following boundary conditions

$$\psi=1: \quad G(f,\bar{f},f',\bar{f}')=0 \tag{2.2}$$

$$\psi=0: \quad \operatorname{Im} f=0 \tag{2.3}$$

It is impossible to solve Problem 1 for an arbitrary function G. However, if the function f is periodic in ψ, Problem 1 reduces to a system of ordinary differential equations. The system consists of an infinite number of equations if the period is an irrational number. If the period is a rational number, the system is finite and Problem 1 can be solved in quadratures.

In fact, according to the symmetry principle, from boundary condition (2.3) we obtain the corollary: $\overline{f(\varphi+i)}=f(\varphi-i)$, and hence condition (2.2) can be rewritten without the conjugation sign

$$G\left(f(\varphi+i), f(\varphi-i), \frac{d}{d\varphi}f(\varphi+i), \frac{d}{d\varphi}f(\varphi-i)\right)=0$$

The last equation holds on the upper boundary of the strip (2.2). However, it can be continued analytically into the strip, since the function f there is analytic. Moreover, this equation can be continued analytically into the region outside the strip, if this region adjoins the strip and has no singular points of the function f. Suppose this region, which also has the form of a strip, perpendicular to strip (2.1), exists. Then we have the differential-difference equation

$$G\left(f(\chi+i), f(\chi-i), \frac{d}{d\chi}f(\chi+i), \frac{d}{d\chi}f(\chi-i)\right)=0 \tag{2.4}$$

which holds in this region.

Introducing the auxiliary functions

$$P_{j+1}(\chi)=f(\chi+2ij) \tag{2.5}$$

and substituting them into Eq. (2.4), we obtain an infinite system of ordinary differential equations

$$G\left(P_{j+1}(\chi), P_j(\chi), \frac{d}{d\chi}P_{j+1}(\chi), \frac{d}{d\chi}P_j(\chi)\right)=0 \tag{2.6}$$

The system becomes finite if the following equality is satisfied

$$f(\chi)=f\left(\chi+i\frac{2n}{m}\right)$$

where m and n are certain integers. In other words, the function $f(\chi)$ has a rational imaginary period. The quantity $i2n$ is also a period, and hence, by definition (2.5), we have

$$P_{n+1}(\chi)=f(\chi+i2n)=f(\chi)=P_1(\chi) \tag{2.7}$$

Consequently, we have from relations (2.6) and (2.7) a system of n ordinary differential equations

$$G(P_{j+1}, P_j, P'_{j+1}, P'_j)=0, \quad P_{n+1}=P_1, \quad j=1,\ldots,n \tag{2.8}$$

Hence, boundary-value problem 1 reduces to the solution of system (2.8). Solving system (2.8) and then putting $f=P_1$, we obtain the solution of boundary-value problem 1.

3. Formulation of the wave problem for waves on the surface of a liquid

We now change to a system of coordinates in which the flow is steady and the liquid flows from left to right (Fig. 2). We will place the origin of the system of coordinates on the bottom, where the X axis is directed along the bottom to the right, while the Y axis is vertically upwards, so that it passes through the crest of the wave (the point B in Fig. 2). Suppose h_0 and u_0 are the depth and velocity at the point B and g is the acceleration due to gravity. We will equate the stream function Ψ to zero on the bottom and put $\Psi=\Psi_0>0$ on the free surface. Then, in the plane of the dimensionless complex potential

$$\chi=\varphi+i\psi=(\Phi+i\Psi)/\Psi_0$$

the regions occupied by the liquid will correspond to a strip of unit width. We need to obtain the function

$$Z=X+iY=h_0 f(\chi)$$

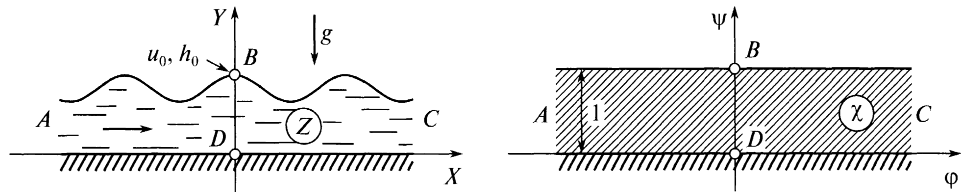

Fig. 2.

which is a conformal mapping of this strip into the region occupied by the liquid. Since this conformal mapping is produced by a non-unique method, the following normalization is necessary

$$f(i) = i, \quad f(0) = 0$$

The Bernoulli integral on the free surface

$$\frac{|V|^2}{2} + gY = \frac{u_0^2}{2} + gh_0$$

after substituting the expression for the modulus of the velocity vector

$$|V|^2 = \left|\frac{d(\Phi + i\Psi)}{d(X + iY)}\right|^2 = \frac{\Psi_0^2}{h_0^2}\left|\frac{d\chi}{df}\right|^2$$

and introducing two dimensionless constants

$$F = u_0/\sqrt{gh_0}, \quad \delta = \Psi_0/(u_0 h_0)$$

can be written in the following dimensionless form

$$\left|\frac{df}{d\chi}\right|^2 = \frac{\delta^2}{1 + 2(1 - \mathrm{Im}f)/F^2} \tag{3.1}$$

We will investigate the periodic waves. However, the function $f(\varphi + i\psi)$ will not be periodic in φ. We must find a constant α such that if the conformal mapping is carried out with the function represented in the form

$$f(\chi) = \alpha\chi + W(\chi) \tag{3.2}$$

the function $W(\varphi + i\psi)$ will be periodic in φ. On the right-hand side of Eq. (3.2) the first term is the conformal mapping of a strip of unit width onto a strip of width α, while the second term – the function $W(\chi)$ – superimposes periodic waves on this strip.

If we substitute expression (3.2) into Bernoulli integral (3.1), we obtain

$$\left|\frac{dW}{d\chi} + \alpha\right|^2 = \frac{\delta^2}{1 + 2(1 - \alpha - \mathrm{Im}W)/F^2} \tag{3.3}$$

We divide the numerator and denominator of the right-hand side here by the quantity

$$\beta = 1 + 2(1 - \alpha)/F^2 \tag{3.4}$$

Then, after introducing two more dimensionless constants

$$q = \frac{\delta^2}{\beta}, \quad \gamma = \frac{1}{F^2\beta} \tag{3.5}$$

the problem of finding gravitational waves on the surface of the liquid reduces to the following form

Problem 2. It is required to find constants α, q, γ and a function $W(\chi)$ of the complex variable $\chi = \varphi + i\psi$, analytic in the strip

$$0 < \psi < 1, \quad -\infty < \varphi < \infty$$

periodic in φ and satisfying the following conditions

$$W(0) = 0, \quad W(i) = i(1 - \alpha) \tag{3.6}$$

$$\psi = 1: \quad \left|\frac{dW}{d\chi} + \alpha\right|^2 = \frac{q}{1 - 2\gamma\mathrm{Im}W} \tag{3.7}$$

$$\psi = 0: \quad \mathrm{Im}W = 0 \tag{3.8}$$

Here (3.6) is a normalization condition, (3.7) is the condition for the pressure to be constant, and (3.8) is the even-bottom condition.

It is well known that this problem has a solution in the form of a two-parameter family of periodic waves, having one crest and one trough per period.

4. The Stokes expansion

To obtain a solution of Problem 2 in the form of series (1.1) all the quantities occurring in the problem must be expanded in powers of ϵ. We have for the function $W(\chi)$

$$W(\chi) = A(\varphi,\psi) + iB(\varphi,\psi) = \epsilon W^{(1)}(\chi) + \epsilon^2 W^{(2)}(\chi) + \ldots$$

The corresponding Stokes expansions for the real and imaginary parts have the form

$$A(\varphi,\psi) = \epsilon A^{(1)}(\varphi,\psi) + \epsilon^2 A^{(2)}(\varphi,\psi) + \ldots,$$
$$B(\varphi,\psi) = \epsilon B^{(1)}(\varphi,\psi) + \epsilon^2 B^{(2)}(\varphi,\psi) + \ldots \quad (4.1)$$

In addition to the expansions for the functions we must also consider similar expansions for the constants. We can use expansions for the constants q and γ, occurring in Problem 2, but it is more convenient to consider expansions for δ and F. We have

$$\alpha = 1 + \epsilon\alpha^{(1)} + \epsilon^2\alpha^{(2)} + \ldots, \quad \delta = 1 + \epsilon\delta^{(1)} + \epsilon^2\delta^{(2)} + \ldots,$$
$$F = F^{(0)} + \epsilon F^{(1)} + \epsilon^2 F^{(2)} + \ldots \quad (4.2)$$

Substituting series (4.1) and (4.2) into Problem 2 and collecting terms of like powers of ϵ, we obtain a series of linear boundary-value problems.

We will formulate the first of these. We rewrite boundary condition (3.3) in the form

$$\alpha^2 + 2\alpha B_\psi + B_\psi^2 + B_\varphi^2 = \frac{\delta^2}{1 + 2(1-\alpha)/F^2 - 2B/F^2}$$

Differentiating this condition, and also the normalization condition (3.6) and the rough-bottom condition (3.8) with respect to ϵ and then letting ϵ tend to zero, we obtain the following boundary-value problem.

Problem 2′. It is required to obtain constants $\alpha^{(1)}$, $\delta^{(1)}$ and $F^{(0)}$ and a function $W^{(1)}(\chi) = A^{(1)}(\varphi,\psi) + iB^{(1)}(\varphi,\psi)$ of the complex variable $\chi = \varphi + i\psi$, analytic in the strip

$$0 < \psi < 1, \quad -\infty < \varphi < \infty$$

periodic in φ and satisfying the following conditions

$$W^{(1)}(0) = 0, \quad W^{(1)}(i) = -i\alpha^{(1)} \quad (4.3)$$

$$\psi = 1: \ \alpha^{(1)} + B_\psi^{(1)} = \delta^{(1)} + (\alpha^{(1)} + B^{(1)})F^{(0)^{-2}} \quad (4.4)$$

$$\psi = 0: \quad B^{(1)} = 0 \quad (4.5)$$

Here (4.3) is a normalization condition, (4.4) is the condition for the pressure to be constant, and (4.5) is the even-bottom condition.

The solution of the homogeneous part of the inhomogeneous boundary-value problem 2′, i.e., the solution of the boundary-value problem with the conditions

$$\psi = 1: \quad B_\psi^{(1)} = B^{(1)}F^{(0)^{-2}} \quad (4.6)$$

$$\psi = 0: \quad B^{(1)} = 0 \quad (4.7)$$

must be sought in the form of the trigonometric function $W^{(1)} = \sin(2\theta\chi)$ or $W^{(1)} = \cos(2\theta\chi)$. Here θ is a certain parameter specifying the wavelength in the plane of the complex potential. We will choose the sine, since the first normalization condition of (4.3) is not satisfied for the cosine. Then

$$B^{(1)} = \mathrm{sh}(2\theta\psi)\cos(2\theta\varphi), \quad B_\psi^{(1)} = 2\theta\,\mathrm{ch}(2\theta\psi)\cos(2\theta\varphi)$$

Substituting these expressions into Eq. (4.6), we obtain the dispersion equation

$$F^{(0)^2} = \frac{\mathrm{th}2\theta}{2\theta} \quad (4.8)$$

We will now obtain a particular solution of the inhomogeneous linear boundary-value problem 2′. It can be seen from condition (4.4) that a particular solution of the form

$$B^{(1)} = 0$$

exists if it is required to satisfy the following equality, relating the constants

$$\alpha^{(1)} = \delta^{(1)} + \alpha^{(1)}F^{(0)^{-2}} \quad (4.9)$$

Using normalization condition (4.3), we obtain

$$\alpha^{(1)} = iW^{(1)}(i) = i\sin(2\theta i) = -\mathrm{sh}2\theta$$

From equalities (4.8) and (4.9) we obtain

$$\delta^{(1)} = 2\theta\,\mathrm{ch}2\theta - \mathrm{sh}2\theta$$

Hence, boundary-value problem 2′ is solved. Collecting all the formulae together, we obtain a two-parameter family of linear sinusoidal waves, which depend on the two parameters ε and θ

$$f(\chi) = \chi(1-\epsilon\,\mathrm{sh}2\theta) + \epsilon\sin 2\theta\chi + O(\epsilon^2),\quad \alpha = 1-\epsilon\,\mathrm{sh}2\theta + O(\epsilon^2)$$

$$\delta = 1+\epsilon(2\theta\,\mathrm{ch}2\theta - \mathrm{sh}2\theta) + O(\epsilon^2),\quad F = \left(\frac{\mathrm{th}2\theta}{2\theta}\right)^{1/2} + O(\epsilon) \tag{4.10}$$

This solution describes small-amplitude waves.

5. Expansion within the framework of shallow-water theory

Non-linear shallow-water theory is the second well-known method for the approximate investigation of Problem 2. The expansion within the framework of this theory is usually constructed using different expansions in two different directions: vertical and horizontal. As a result of this operation the functions of the complex variable present in the problem lose their analyticity. If it is necessary to use a conformal mapping, carried out on the function (3.2), one must proceed differently. For long waves $W(\chi)$ is a slowly varying function of χ. This means that a small parameter ε exists and W must be considered as a function of the argument $z = x + iy = \varepsilon\chi$.

Note that $\varepsilon = 0$ corresponds to the liquid motion with constant velocity, and the velocity cannot be arbitrary – the Froude number must be equal to unity. Only in this case will the expansion constructed below not be contradictory.

Hence, for the unknown function $W(z)$ we have the following boundary-value problem in a strip of width ε.

Problem 3. It is required to obtain constants α, q and γ and a function $W(z) = A(x,y) + iB(x,y)$ of the complex variable $z = x + iy$, analytic in the strip

$$0 < y < \varepsilon,\quad -\infty < x < \infty$$

periodic in x and which satisfies the following conditions

$$W(0) = 0,\quad W(i\varepsilon) = i(1-\alpha) \tag{5.1}$$

$$y = \varepsilon:\quad \left|\varepsilon\frac{dW}{dz} + \alpha\right|^2 = \frac{q}{1-2\gamma\,\mathrm{Im}\,W} \tag{5.2}$$

$$y = 0:\quad \mathrm{Im}\,W = 0 \tag{5.3}$$

Here (5.1) is a normalization condition, (5.2) is the condition of constant pressure, and (5.3) is the even-bottom condition.

Investigating this problem in the form (1.2), we arrive at the series

$$W(z) = \varepsilon W^{(0)}(z) + \varepsilon^3 W^{(1)}(z) + \ldots$$

or at analogous series for the real and imaginary parts

$$A(x,y) = \varepsilon A^{(0)}(x,y) + \varepsilon^3 A^{(1)}(x,y) + \ldots,\quad B(x,y) = \varepsilon B^{(0)}(x,y) + \varepsilon^3 B^{(1)}(x,y) + \ldots \tag{5.4}$$

It is also necessary to seek the constants in the form of similar asymptotic expansions

$$\alpha = 1 + \varepsilon^2\alpha^{(1)} + \varepsilon^4\alpha^{(2)} + \ldots,\quad \delta = 1 + \varepsilon^2\delta^{(1)} + \varepsilon^4\delta^{(2)} + \ldots,$$

$$F = 1 + \varepsilon^2 F^{(1)} + \varepsilon^4 F^{(2)} + \ldots \tag{5.5}$$

Boundary-value problem 2 in the limit as $\varepsilon \to 0$ also converts into boundary-value problem 2′. This does not occur for Problem 3, its distinguishing feature being degeneration as $\varepsilon \to 0$. In fact, in this case the width of the strip in the z plane approaches zero, i.e., the upper and lower boundaries coincide, and in the limit as $\varepsilon \to 0$ we obtain only certain differential equations and not a boundary-value problem.

To derive these equations we substitute expansions (5.4) and (5.5) into boundary conditions (5.2) and (5.3) and collect terms of like powers of ε. We first expand the right-hand side of boundary condition (5.2) in a power series in powers of $B = \mathrm{Im}W$. As a result we obtain the constant-pressure condition in the following form

$$\alpha^2 + 2\varepsilon\alpha B_y + \varepsilon^2(B_x^2 + B_y^2) = q(1 + 2\gamma B + 4\gamma^2 B^2 + \ldots) \tag{5.6}$$

The constants q and γ here must be substituted in the form of series, where it follows from equalities (3.5) that the first terms in these series must be equal to unity:

$$q = 1 + \varepsilon^2 q^{(1)} + \varepsilon^4 q^{(2)} + \ldots,\quad \gamma = 1 + \varepsilon^2\gamma^{(1)} + \varepsilon^4\gamma^{(2)} + \ldots$$

Boundary condition (5.6) is satisfied when $y = \varepsilon$, i.e., small quantities occur not only in the equation but also in the arguments of the unknown functions. In order to expand the functions $B(x,\varepsilon)$, $B_x(x,\varepsilon)$, $B_y(x,\varepsilon)$ in series in powers of ε we must obtain corresponding expansions for $B^{(j)}(x, \varepsilon)$, $B_x^{(j)}(x, \varepsilon)$, $B_y^{(j)}(x, \varepsilon)$ for all j and only then substitute them into equalities (5.4). Using a Taylor series we have

$$B^{(j)}(x,\varepsilon) = \varepsilon B_y^{(j)} + \varepsilon^3 \frac{B_{yyy}^{(j)}}{3!} + \dots$$
$$B_x^{(j)}(x,\varepsilon) = \varepsilon B_{yx}^{(j)} + \varepsilon^3 \frac{B_{yyyx}^{(j)}}{3!} + \dots, \quad B_y^{(j)}(x,\varepsilon) = B_y^{(j)} + \varepsilon^2 \frac{B_{yyy}^{(j)}}{2!} + \dots \tag{5.7}$$

The arguments of the functions on the right-hand sides of these equalities are not written out. We will agree henceforth that if the arguments are not written out, the values of the functions are taken when $y = 0$, which corresponds to the bottom.

When writing formulae (5.7) we took into account that all the even derivatives with respect to y when $y = 0$ are equal to zero. This is a consequence of the even-bottom condition (5.3), i.e., $B^{(j)} = 0$ for all j, and also a consequence of the fact that the functions $B^{(j)}(x,y)$ are harmonic, i.e., $B_{xx}^{(j)} + B_{yy}^{(j)} = 0$.

Substituting expressions (5.7) into series (5.4), we obtain expansions for the function B and its derivatives on the free surface

$$B(x,\varepsilon) = \varepsilon^2 B_y^{(0)} + \varepsilon^4\left(\frac{B_{yyy}^{(0)}}{3!} + B_y^{(1)}\right) + O(\varepsilon^6)$$
$$B_x(x,\varepsilon) = \varepsilon^2 B_{yx}^{(0)} + \varepsilon^4\left(\frac{B_{yyyx}^{(0)}}{3!} + B_{yx}^{(1)}\right) + O(\varepsilon^6)$$
$$B_y(x,\varepsilon) = \varepsilon B_y^{(0)} + \varepsilon^3\left(\frac{B_{yyy}^{(0)}}{2!} + B_y^{(1)}\right) + O(\varepsilon^5) \tag{5.8}$$

We now substitute series (5.8) obtained into condition (5.6) and collect terms of like powers of ε. For ε^2 and ε^4 respectively this gives the following equations

$$2\alpha^{(1)} - q^{(1)} = 0$$
$$\frac{2}{3}B_{yyy}^{(0)} - 2B_y^{(0)}\left(\alpha^{(1)} + \gamma^{(1)}\right) - 3B_y^{(0)2} + 2\alpha^{(2)} + \alpha^{(1)^2} - q^{(2)} = 0 \tag{5.9}$$

It is convenient to retain one unknown function $u(x) = A_x^{(0)}(x, 0)$. Taking into account the Cauchy–Riemann relations

$$A_x^{(0)} = B_y^{(0)}, \quad A_y^{(0)} = -B_x^{(0)}$$

we can express all the derivatives, occurring in the second equation of (5.9), in terms of derivatives of $u(x)$. Hence, to obtain long waves we must solve the following problem.

Problem 3′. It is required to obtain constants $\alpha^{(1)}, q^{(1)}, \gamma^{(1)}$ and the real function $u(x)$, periodic in x and satisfying the following conditions

$$u(0) = -\alpha^{(1)}, \quad u_x(0) = 0 \tag{5.10}$$
$$2\alpha^{(1)} - q^{(1)} = 0 \tag{5.11}$$
$$-\frac{2}{3}u_{xx} - 2u(\alpha^{(1)} + \gamma^{(1)}) - 3u^2 + (\alpha^{(1)})^2 + 2\alpha^{(2)} - q^{(2)} = 0 \tag{5.12}$$

Here the first condition of (5.10) is a normalization condition, the second is a symmetry condition, and (5.11) and (5.12) are constant-pressure conditions.

We will explain the origin of conditions (5.10). If the second normalization condition (5.1) is differentiated with respect to ε and we then take the limit as $\varepsilon \to 0$, this gives $B_y^{(0)}(0,0) = -\alpha^{(1)}$. Taking into account the fact that

$$B_y^{(0)}(0,0) = A_x^{(0)}(0,0) = u(0) \tag{5.13}$$

we obtain the first condition of (5.10). The second condition of (5.10) is obtained from the condition for the gravitational waves to be symmetrical about the vertical axis, passing through the crest of the wave. This means that when $y = 0$ the function $W(z)$ must be pure imaginary, i.e., $A(0,y) = 0$ for all y. Consequently, $A_{yy}(0,y) = 0$. Taking the limit in the last equality as $\varepsilon \to 0$, we obtain $A_{yy}^{(0)}(0, y) = 0$. Further, taking into account the fact that the function is harmonic, we have the equality $A_{xx}^{(0)}(0, y) = 0$, which holds for all y, including when $y = 0$. Taking the limit in this equality as $y \to 0$, we obtain the second condition of (5.10).

In addition to the constants $\alpha^{(1)}, q^{(1)}, \gamma^{(1)}$, which must be obtained, Problem 3′ contains one other constant $2\alpha^{(2)} - q^{(2)}$ in Eq. (5.12). A feature of Problem 3′ is the fact that this constant, generally speaking, is not defined.

Equation (5.12) is easily integrated once. As a result we have a second-order differential equation

$$\frac{1}{3}u_x^2 = P(u) \tag{5.14}$$

on the right-hand side of which there is a third-degree polynomial, containing one more real constant of integration C:

$$P(u) = -u^3 - \left(\alpha^{(1)} + \gamma^{(1)}\right)u^2 + \left(2\alpha^{(2)} + \alpha^{(1)^2} - q^{(2)}\right)u + C \tag{5.15}$$

It follows from conditions (5.10) that one of the roots of polynomial $P(u)$ must be $-\alpha^{(1)}$. We will denote the other two by u_1 and u_2. We have

$$P(u) = -(u + \alpha^{(1)})(u - u_1)(u - u_2) \tag{5.16}$$

The roots u_1 and u_2 are related to the previously derived constants as follows:

$$(u - u_1)(u - u_2) = u^2 + \gamma^{(1)}u - \alpha^{(1)}(\gamma^{(1)} + \alpha^{(1)}) - (2\alpha^{(2)} - q^{(2)}) \tag{5.17}$$

This relation is obtained by dividing polynomial (5.15) by the monomial $(u+\alpha^{(1)})$.

We will show that the roots u_1 and u_2 are real. Since we are seeking periodic waves, the velocity on the bottom under a wave crest reaches a minimum, while under a trough it reaches a maximum. The quantity $(df/d\chi)^{-1}$ has the meaning of the complex velocity, and on the bottom it takes a real value. We therefore conclude from the condition for this quantity to have an extremum that on the bottom there are at least two points (under a crest and under a trough), where $d^2f/d\chi^2 = 0$, while the quantity $df/d\chi$ at these points is real and different. Changing from χ to z and from the function $f(\chi)$ to the function $W(z)$, we conclude that on the bottom (with $z = x$) there are two points where $d^2W/dz^2 = 0$, while the derivative dW/dz at these points is real and different. For small ε the main contribution to the solution is made by the function $W^{(0)}(z)$, and hence it satisfies the same periodicity conditions as $W(z)$. Consequently, on the bottom for real z there must be at least two points where the quantity $dW^{(0)}/dz$ is real and different while $d^2W^{(0)}/dz^2 = 0$. Since on the bottom when $z = x$ we have $dW^{(0)}/dz = u$, while $d^2W^{(0)}/dz^2 = u_x$, we arrive at the assertion that u_x vanishes for two different real values of u. This means that the polynomial $P(u)$ has two real roots. Hence, by virtue of the fact that the coefficients $P(u)$ are real, it follows that the third root is also real.

For the further integration of Eq. (5.14) it now remains to solve the problem of choosing the sign of u_x when extracting the root in this equation. The derivative $df/d\chi$ is a quantity which is the inverse of the velocity, and for small ε it is approximately equal to $\alpha + \varepsilon u$ on the bottom. Since the velocity increases from the crest to the trough, the quantity u must decrease. The point under the crest corresponds to $x = 0$, and hence we conclude that $u_x < 0$ when $x > 0$.

Consequently, when extracting the root we must take the minus sign, while if we take into account the normalization (the first condition (5.10)), we have, from Eqs. (5.14) and (5.16)

$$-\frac{1}{\sqrt{3}} \int\limits_{-\alpha^{(1)}}^{u} \frac{du}{\sqrt{-(u + \alpha^{(1)})(u - u_1)(u - u_2)}} = x, \quad x \geq 0 \tag{5.18}$$

We will consider the problem of the mutual position of the three roots of the polynomial $P(u)$. Without loss of generality we will assume that $u_2 > u_1$. Then, the three possible cases of the position of the roots of $P(u)$ are shown in Fig. 3. The maximum value of u is reached under the crest, and, as a consequence of the first condition of (5.10), it is equal to $-\alpha^{(1)}$. When $u < -\alpha^{(1)}$ we must have $P(u) > 0$. Hence we see that the position of the roots, shown in Fig. 3a, is inadmissible. The case shown in Fig. 3b is also impossible, since a value of u, less than $-\alpha^{(1)}$, must exist for which $P(u) = 0$. The correct mutual position of the roots is shown in Fig. 3c. Part of the graph of the function $P(u)$, used for integration inequality (5.18), is shown by the heavy line.

Consequently $u_1 < u_2 < -\alpha^{(1)}$, and we can then, in integral (5.18), make the change of integration variable

$$u = -\alpha^{(1)} + (\alpha^{(1)} + u_2)\sin^2\Omega \tag{5.19}$$

after which equality (5.18) takes the form

$$\frac{\mathcal{F}(\Omega, k)}{\rho} = x \tag{5.20}$$

where $\mathcal{F}(\Omega, k)$ is the elliptic integral of the second kind of modulus k, while ρ is a constant where

$$k = \sqrt{\frac{\alpha^{(1)} + u_2}{\alpha^{(1)} + u_1}}, \quad \rho = \frac{\sqrt{3}}{2}\sqrt{-\alpha^{(1)} - u_1} \tag{5.21}$$

Taking the definition of the Jacobi elliptic sine of modulus k into account

$$\mathrm{sn}\mathcal{F} = \sin\Omega$$

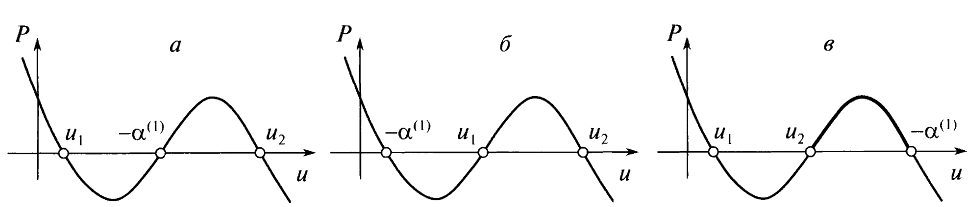

Fig. 3.

we obtain from formulae (5.19) and (5.20)

$$u(x) = -\alpha^{(1)} + (\alpha^{(1)} + u_2)\mathrm{sn}^2(\rho x) \quad (5.22)$$

Hence, Problem 3′ is partially solved – we have obtained the required function $u(x)$. It remains to find the unknown constants. We will first extend equality (5.22) analytically from the real axis $z = x$ into the region of complex values of z. Since there are analytical functions of z on the left and right-hand sides of equality (5.22), then, extending them analytically, we have

$$W_z^{(0)} = -\alpha^{(1)} + (\alpha^{(1)} + u_2)\mathrm{sn}^2(\rho z)$$

Integrating this equation and replacing z by $\varepsilon\chi$, we obtain, in the long-wave approximation, a formula for the conformal mapping of the strip into the χ plane occupied by the liquid

$$f(\chi) = \alpha\chi + W = (1 + \varepsilon^2\alpha^{(1)} + \ldots)\chi + \varepsilon\left(-\alpha^{(1)}\varepsilon\chi + (\alpha^{(1)} + u_2)\int_0^{\varepsilon\chi}\mathrm{sn}^2(\rho z)dz\right) + \ldots =$$

$$= \chi + \varepsilon^2(\alpha^{(1)} + u_2)\int_0^{\chi}\mathrm{sn}^2(\varepsilon\rho\xi)d\xi + \ldots$$

It follows from formulae (5.21) that $\alpha^{(1)} + u_2 = -4k^2\rho^2/3$. Hence, introducing the new small parameter $\theta = \varepsilon\rho$, we finally obtain

$$f(\chi) = \chi - \frac{4}{3}\theta^2 k^2\int_0^{\chi}\mathrm{sn}^2(\theta\xi)d\xi + O(\theta^4) \quad (5.23)$$

Now, using this formula, it is necessary to choose the constant α so that the function $f(\chi) - \alpha\chi$ is periodic in $\varphi = \mathrm{Re}\chi$. We will define the function

$$E(\zeta) = \int_0^{\zeta}\mathrm{dn}^2\xi d\xi; \quad \mathrm{dn}\xi = \sqrt{1 - k^2\mathrm{sn}^2\xi} \quad (5.24)$$

Solution (5.23) can be chosen in terms of the function (5.24)

$$W(\chi) = f(\chi) - \alpha\chi \approx \chi\left(1 - \frac{4}{3}\theta^2 - \alpha\right) + \frac{4}{3}\theta E(\theta_\chi) \quad (5.25)$$

We know,[4] that the function $\mathbf{E}(\zeta)$ is not periodic, but, when the argument is increased by $2\mathbf{K}$ it obtains a constant increment $2\mathbf{E}$. Here $\mathbf{K}(k)$ and $\mathbf{E}(k)$ are complete elliptic integrals of the first and second kind in the modulus k. Hence, after an increment by $2\mathbf{K}/\theta$ in the argument χ in formula (5.25), we obtain

$$W\left(\chi + \frac{2\mathbf{K}}{\theta}\right) \approx \left(\chi + \frac{2\mathbf{K}}{\theta}\right)\left(1 - \frac{4}{3}\theta^2 - \alpha\right) + \frac{4}{3}\theta[E(\theta\chi) + 2\mathbf{E}]$$

Requiring that the function $W(\chi)$ has not changed, we obtain

$$\alpha = 1 + \frac{4}{3}\theta^2\left(\frac{\mathbf{E}}{\mathbf{K}} - 1\right) + O(\theta^4)$$

We will obtain the Froude number $F^{(1)}$. Differentiating with respect to ε and taking the limit as $\varepsilon \to 0$ in equality (3.4) and in the second formula of (3.5) and using Vieta's theorem for quadratic trinomial (5.17), we obtain, respectively,

$$\beta^{(1)} = -2\alpha^{(1)}, \quad 2F^{(1)} = -\gamma^{(1)} - \beta^{(1)}, \quad u_1 + u_2 = -\gamma^{(1)} \quad (5.26)$$

Hence it follows that

$$2F^{(1)} = (\alpha^{(1)} + u_1) + (\alpha^{(1)} + u_2)$$

Further, using initially the first formula of (5.21) and then the second formula, we obtain

$$2F^{(1)} = (1 + k^2)(\alpha^{(1)} + u_1) = -\frac{4}{3}(1 + k^2)\rho^2$$

We can similarly fund $\delta^{(1)}$. Differentiating with respect to ε and taking the limit as $\varepsilon \to 0$ in the first formula of (3.5), we obtain

$$2\delta^{(1)} = q^{(1)} + \beta^{(1)}$$

Further, using condition (5.11) successively and the first formula of (5.26), we obtain

$$2\delta^{(1)} = 2\alpha^{(1)} + \beta^{(1)} = -\beta^{(1)} + \beta^{(1)} = 0$$

Thus, collecting it all together, we obtain a two-parameter family of long waves on the liquid surface, called cnoidal waves; they depend on the real parameters k and θ ($0 \le k \le 1$, θ is a small positive parameter) and they are given by the formulae

$$f(\chi) = \chi - \frac{4}{3}\theta^2 k^2 \int_0^{\chi} \mathrm{sn}^2(\theta\xi)d\xi + O(\theta^4), \quad \alpha = 1 + \frac{4}{3}\theta^2\left(\frac{\mathbf{E}}{\mathbf{K}} - 1\right) + O(\theta^4)$$

$$\delta = 1 + 0\theta^2 + O(\theta^4), \quad F = 1 - \frac{2}{3}(1 + k^2)\theta^2 + O(\theta^4) \tag{5.27}$$

6. Comparison of the two solutions

Thus, we have obtained two solutions (4.10) and (5.27), each of which holds in its own region, shown in Fig. 1. However, these regions intersect, and hence it is interesting to compare the solutions in the regions where they intersect, i.e., to consider the long-wave limit in the Stokes expansion (4.10) and the limit of very small amplitudes in expansion (5.27).

In formulae (4.10) the long-wave limit corresponds to taking the limit as $\theta \to 0$. This gives

$$f(\chi) \simeq \chi(1 - \epsilon 2\theta) + \epsilon \sin 2\theta_{\chi}, \quad \alpha \simeq 1 - \epsilon 2\theta, \quad \delta \simeq 1, \quad F \simeq 1 - \frac{2}{3}\theta^2 \tag{6.1}$$

Naturally, no solitary waves are obtained here.

Expansion (5.27) holds in a small neighbourhood of the point C in Fig. 1. Since the neighbourhood is small, the wave amplitude there is extremely small. Hence, in the long-wave expansion, smallness of the amplitude is imposed from the beginning. However, if it is necessary to take the limit of very small amplitudes, one must consider taking the limit as $k \to 0$. In fact, the angle between the radius vector, drawn from the point C, and the negative direction of the abscissa axis depends on the parameter k. When $k = 0$ this angle is equal to zero. When $k \to 0$ we have a transition from sn to sin. Also taking into account the expansion

$$\frac{\mathbf{E}}{\mathbf{K}} = 1 - \frac{1}{2}k^2 + \ldots \tag{6.2}$$

we obtain from relations (5.27)

$$f(\chi) \simeq \chi\left(1 - \frac{2}{3}\theta^2 k^2\right) + \frac{1}{3}\theta k^2 \sin 2\theta\chi, \quad \alpha \simeq 1 - \frac{2}{3}\theta^2 k^2, \quad \delta \simeq 1, \quad F \simeq 1 - \frac{2}{3}\theta^2 \tag{6.3}$$

i.e., we have sinusoidal waves. It can be seen that expansions (4.10) and (5.27) may coalesce. In fact, formulae (6.1) and (6.3) are identical if the parameters are related by the relation $\epsilon = \theta k^2/3$.

When $k = 1$ the angle between the radius-vector, drawn from the point C, and the negative direction of the abscissa axis reaches a maximum. Hence, the limit as $k \to 1$ in formulae (5.27) correspond to solitary waves. In this case sn changes into th and, as a result, we obtain

$$f(\chi) \simeq \chi\left(1 - \frac{4}{3}\theta^2\right) + \frac{4}{3}\theta \,\mathrm{th}\theta\chi, \quad \alpha \simeq 1 - \frac{4}{3}\theta^2, \quad \delta \simeq 1, \quad F \simeq 1 - \frac{4}{3}\theta^2 \tag{6.4}$$

The conformal mapping contains a hyperbolic tangent, and hence solution (6.4) describes flow with a free surface, having a local elevation.

7. A subsidiary problem

Two solutions were obtained above: the Stokes expansion (4.10) and the long-wave expansion (5.27), neither of which is entirely suitable for all wavelengths. At the same time, each of these solutions is obtained in the small wave amplitude approximation. Hence, the idea arises of considering the case of small amplitudes, neglecting terms of higher order of infinitesimals, while retaining quadratic terms.

We will rewrite condition (3.7) for the pressure to be constant in the form

$$\left|\frac{dW}{d\chi} + \alpha\right|^2 = q\left[1 + 2\gamma \mathrm{Im} W + (2\gamma \mathrm{Im} W)^2 + (2\gamma \mathrm{Im} W)^3 + \ldots\right] \tag{7.1}$$

For small-amplitude waves the quantity ImW is small, and hence if we cut off the series on the right-hand side, we obtain an approximate solution suitable for small amplitudes. In the linear boundary-value problem 2′ we dropped all terms in the square brackets, apart from the first two. We will now retain the third term and, instead of Problem 2, we will solve the boundary-value problem with quadratic nonlinearity.

Problem 4. It is required to obtain the constants α, q and γ and the function $W(\chi)$, periodic in φ and analytic in the strip

$$0 < \psi < 1, \quad -\infty < \varphi < \infty$$

and which satisfies the following conditions

$$W(0) = 0, \quad W(i) = i(1 - \alpha) \tag{7.2}$$

$$\psi = 1: \quad \left|\frac{dW}{d\chi} + \alpha\right|^2 = q\left[1 + 2\gamma \mathrm{Im} W + (2\gamma \mathrm{Im} W)^2\right] \tag{7.3}$$

$$\psi = 0: \quad \operatorname{Im} W = 0 \tag{7.4}$$

The approximate condition for the pressure to be constant (7.3) appeared for the first time in Ovsyannikov's paper,[5] in which the solution of Problem 4 for solitary waves was obtained. Here we obtain an extension of this solution to the case of periodic gravitational waves.

Note an important property of Problem 4. If the solution is sought in the form of series (1.1) and (1.2), we obtain a result which is completely identical with the result represented by formulae (4.10) and (5.27). In other words, the first terms of the Stokes expansion and the long-wave expansion are identical for Problem 2 and for Problem 4. This conclusion is obvious, since when obtaining relations (4.10) and (5.27) the cubic terms and terms of higher order of infinitesimals were ignored. Hence, Problem 4 contains both solitary waves and linear sinusoidal waves. Hence, the exact solution of Problem 4 is uniformly suitable for all wavelengths, and it should describe quite well small-amplitude waves in the case of both short and long waves.

However Problem 4 is non-linear. We will attempt to solve it using the same method used to solve Problem 1. We will denote the function on the right-hand side of boundary condition (7.3) by $S(\operatorname{Im}W)$. We can then rewrite boundary condition (7.3) in the following form

$$\left(\frac{dW(\varphi+i)}{d\varphi}+\alpha\right)\left(\frac{d\overline{W(\varphi+i)}}{d\varphi}+\alpha\right) = S\left(\frac{W(\varphi+i)-\overline{W(\varphi+i)}}{2i}\right) \tag{7.5}$$

Introducing the functions $P_1(\chi)$,$P_2(\chi)$,... by the rule

$$P_j(\chi) = W(\chi + i(2j-2)) \tag{7.6}$$

we obtain

$$P_j(\varphi + i(1-2j)) = W(\varphi - i) = \overline{W(\varphi+i)}$$

$$P_{j+1}(\varphi + i(1-2j)) = W(\varphi+i)$$

Consequently, Eq. (7.5) can be written in the form

$$\left(\frac{dP_{j+1}(\varphi+i(1-2j))}{d\phi}+\alpha\right)\left(\frac{dP_j(\varphi+i(1-2j))}{d\varphi}+\alpha\right)$$
$$= S\left(\frac{P_{j+1}(\varphi+i(1-2j)) - P_j(\varphi+i(1-2j))}{2i}\right)$$

or, using the fact that the functions $P_j(\chi)$ are analytic, in the form

$$\left(\frac{dP_{j+1}}{d\chi}+\alpha\right)\left(\frac{dP_j}{d\chi}+\alpha\right) = S\left(\frac{P_{j+1}-P_j}{2i}\right)$$

Hence, Problem 4 reduces to an infinite, generally speaking, system of ordinary differential equations

$$\left(\frac{dP_{j+1}}{d\chi}+\alpha\right)\left(\frac{dP_j}{d\chi}+\alpha\right) = q\left[1 - i\gamma(P_{j+1}-P_j) - \gamma^2(P_{j+1}-P_j)^2\right] \tag{7.7}$$

However, if the solutions of this system are periodic functions with an imaginary period and a modulus of the period equal to a rational number, then, as shown above, the system is finite. System (7.7) is easily solved when it consists of three or four equations.

Further, basing ourselves on these solutions, we were able to guess the structure of the solution in the general case. Note that the solution of Problem 4 obtained turned out to be very similar to the solution of the problem in the long-wave approximation (5.27).

Theorem. The solution of Problem 4 is given by the formula

$$W(\chi) = \mu\chi + \eta\int_0^{\chi} \operatorname{sn}^2\theta\xi\, d\xi \tag{7.8}$$

where μ and η are certain constants.

The proof of the theorem consists of substituting solution (7.8) into differential equation (7.7). We will show that one can always choose constants μ and η such that Eq. (7.7) is satisfied identically. We will use three identities of the theory of elliptic functions

$$\operatorname{sn}^2(u+\upsilon) + \operatorname{sn}^2(u-\upsilon) =$$
$$= 2\frac{\operatorname{sn}^2u + \operatorname{sn}^2\upsilon(k^2\operatorname{sn}^4u - 2k^2\operatorname{sn}^2u - 2\operatorname{sn}^2u + 1) + k^2\operatorname{sn}^4\upsilon\operatorname{sn}^2u}{\Lambda^2(u,\upsilon)} \tag{7.9}$$

$$\operatorname{sn}(u+\upsilon)\operatorname{sn}(u-\upsilon) = \frac{\operatorname{sn}^2u - \operatorname{sn}^2\upsilon}{\Lambda(u,\upsilon)} \tag{7.10}$$

$$\mathrm{sn}^2(u+\upsilon)-\mathrm{sn}^2(u-\upsilon)=2\frac{d}{d\upsilon}\frac{\mathrm{sn}u\,\mathrm{cn}u\,\mathrm{dn}u\,\mathrm{sn}^2\upsilon}{\Lambda^2(u,\upsilon)}$$

$$\Lambda(u,\upsilon)=1-k^2\mathrm{sn}^2u\,\mathrm{sn}^2\upsilon \tag{7.11}$$

Identities (7.9) and (7.10) are well known and are given in all the handbooks on the theory of elliptic functions. The third identity (7.11) is less known. It was used in Toda's book[6] on the investigation of non-linear lattices and was given in a somewhat different form in Whitham's monograph (Ref. 7, Section 17.13).

Using definition (7.6), we obtain from (7.8)

$$P_j(\chi)=\mu[\chi+i(2j-2)]+\eta\int_0^{\chi+i(2j-2)}\mathrm{sn}^2\theta\xi\,d\xi$$

$$P_{j+1}(\chi)=\mu[\chi+i2j]+\eta\int_0^{\chi+i2j}\mathrm{sn}^2\theta\xi\,d\xi$$

Making a replacement of variables in these integrals and changing to an integration variable t using the formula $t=\theta(\xi+i)$ for the first integral and $t=\theta(\xi-i)$ for the second, we obtain

$$P_j(\chi)=\mu[\chi+i(2j-2)]+\frac{\eta}{\theta}\int_u^{\upsilon}\mathrm{sn}^2(t-u)dt,$$

$$P_{j+1}(\chi)=\mu[\chi+i2j]+\frac{\eta}{\theta}\int_{-u}^{\upsilon}\mathrm{sn}^2(t+u)dt \tag{7.12}$$

Here we have introduced the following notation for the constant u and the variable υ

$$u=i\theta,\quad \upsilon=\theta(\chi+i(2j-1)) \tag{7.13}$$

Differentiating (7.12), we obtain

$$\frac{dP_j}{d\chi}=\mu+\eta\,\mathrm{sn}^2(u-\upsilon),\quad \frac{dP_{j+1}}{d\chi}=\mu+\eta\,\mathrm{sn}^2(u+\upsilon)$$

We substitute these expressions into the left-hand side of Eq. (7.7), and it can then be represented in the form

$$\rho^2+\rho\eta[\mathrm{sn}^2(u+\upsilon)+\mathrm{sn}^2(u-\upsilon)]+\eta^2[\mathrm{sn}(u+\upsilon)\mathrm{sn}(u-\upsilon)]^2 \tag{7.14}$$

where

$$\rho=\mu+\alpha \tag{7.15}$$

Using formulae (7.9) and (7.10) we see that the left-hand side of Eq. (7.7), i.e., the function (7.14), multiplied by $\Lambda^2(u,\upsilon)$, is a quadratic trinomial of the variable $\mathrm{sn}^2\upsilon$.

We will show that the right-hand side of Eq. (7.7) also possesses a similar property. It follows from equalities (7.12) that (using formula (7.11))

$$P_{j+1}-P_j=\kappa+\frac{2\eta}{\theta}\frac{\mathrm{sn}u\,\mathrm{cn}u\,\mathrm{dn}u\,\mathrm{sn}^2\upsilon}{\Lambda(u,\upsilon)} \tag{7.16}$$

Here we have introduced a new constant

$$\kappa=2\left(i\mu+\frac{\eta}{\theta}\int_0^u\mathrm{sn}^2\xi\,d\xi\right) \tag{7.17}$$

Substituting expressions (7.16) into the right-hand side of (7.7) and multiplying it by $\Lambda^2(u,\upsilon)$, we again obtain a quadratic trinomial in $\mathrm{sn}^2\upsilon$.

Now equating coefficients of like powers of $\mathrm{sn}^2\upsilon$ in (7.7), multiplied by $\Lambda^2(u,\upsilon)$, we obtain three algebraic equations connecting the constants:

$$(\rho+\eta\,\mathrm{sn}^2u)^2=q(1-i\gamma\kappa-\gamma^2\kappa^2) \tag{7.18}$$

$$\rho(3k^2\mathrm{sn}^4u-2k^2\mathrm{sn}^2u-2\mathrm{sn}^2u+1)+\eta(k^2\mathrm{sn}^6u-\mathrm{sn}^2u)$$
$$=-q(i\gamma+2\gamma^2\kappa)\frac{\mathrm{sn}u\,\mathrm{cn}u\,\mathrm{dn}u}{\theta} \tag{7.19}$$

$$4\rho(k^4\mathrm{sn}^6u - k^4\mathrm{sn}^4u - k^2\mathrm{sn}^4u + k^2\mathrm{sn}^2u) + \eta(1 + k^4\mathrm{sn}^8u - 2k^2\mathrm{sn}^4u)$$
$$= -4\eta q\gamma^2\left(\frac{\mathrm{sn}u\,\mathrm{cn}u\,\mathrm{dn}u}{\theta}\right)^2 \tag{7.20}$$

Normalization condition (7.2) gives one other equation

$$i\mu + \frac{\eta}{\theta}\int_0^u \mathrm{sn}^2\xi\, d\xi = i(1-\alpha) \tag{7.21}$$

and the condition for the function $W(\chi)$ to be periodic

$$\mu\mathbf{K} + \frac{\eta}{k^2}(\mathbf{K} - \mathbf{E}) = 0 \tag{7.22}$$

Hence, if we are given the parameters of a gravitational wave, i.e., we know the constants k and θ, which determine the wavelength and amplitude, then, to obtain the five remaining constants α, q, γ, μ and η, occurring in the solution, we have five algebraic equations (7.18)–(7.22). The system also contains the constants u, ρ and κ, which are found from relations (7.13), (7.15) and (7.17).

We will analyse the system. If the parameters k and θ are specified, it follows from (7.21) and (7.22) that μ and η are linear functions of α. Taking this fact into account, we can rewrite Eqs. (7.18)–(7.20) in the form

$$(A + B\alpha)^2 = q(1 - i\gamma\kappa - \gamma^2\kappa^2), \quad C + D\alpha = q(i\gamma + 2\gamma^2\kappa), \quad M + N\alpha = q\gamma^2\kappa \tag{7.23}$$

Here A, B, C, D, M and N are certain known constants. If we add to Eqs. (7.23) the corollary from Eqs. (7.17) and (7.21)

$$\kappa = 2i(1-\alpha) \tag{7.24}$$

this gives, as a result, a system of four equations for finding the four unknowns q, γ, κ and α. It can be shown that this system reduces to a cubic equation in α, which always has a real root. Hence it follows that system (7.18)–(7.22) also always has solutions. The theorem is proved.

8. The limit solution as $k \to 0$

Solution (7.8) contains elliptic functions and is similar to the formulae describing cnoidal waves. We will show that as $k \to 0$ it gives a Stokes expansion. To do this we will investigate the behaviour of the solution of system (7.18)–(7.22) as $k \to 0$.

When $k \to 0$ we obtain from Eq. (7.20)

$$\eta = \eta q\gamma^2\left(\frac{\mathrm{sh}\,2\theta}{\theta}\right)^2$$

Here the necessary solution is $\eta = 0$. Hence, η is an infinitesimal quantity when $k \to 0$. It is more convenient to seek an expansion of the solution in this quantity.

Taking expansion (6.2) into account, from periodicity condition (7.22) we obtain

$$2\mu + \eta = 0 \tag{8.1}$$

Hence, the quantity μ is also an infinitesimal and of the same order of infinitesimals as η.

Taking formula (8.1) into account, solution (7.8) can be written in the form

$$W(\chi) = \mu\chi + \eta\int_0^\chi \sin^2\theta\xi\, d\xi = -\frac{\eta}{4\theta}\sin 2\theta\chi$$

The normalization condition (7.21) gives, as $k \to 0$,

$$1 - \alpha = -\eta\frac{\mathrm{sh}\,2\theta}{4\theta} \tag{8.2}$$

Hence, $1 - \alpha$ is an infinitesimal quantity and, consequently, the limit value of α is equal to unity.

The limit values of κ and ρ, as follows from relations (7.24) and (7.14), are as follows: $\kappa = 0$ and $\rho = 1$. Consequently, from (7.18) we obtain the limit value

$$q = 1 \tag{8.3}$$

Taking this into account and taking the limit as $k \to 0$ in (7.19), we obtain the limit value

$$\gamma = \frac{2\theta}{\mathrm{th}\,2\theta} \tag{8.4}$$

From relation (3.4) we have the limit value

$$\beta = 1 \tag{8.5}$$

Hence, by the second formula of (3.5) the limit value of the square of the Froude number is found from the formula $F^2 = 1/\gamma$.

We will obtain δ. We have from the first formula of (3.5) the equality

$$\delta^2 = \beta q \tag{8.6}$$

in which we must substitute the expansion for β and q, but taking into account terms of higher order of infinitesimals than in relations (8.5) and (8.3). We will obtain these expansions. Substituting expression (8.2) and the limit value of the square of the inverse Froude number (8.4) into expression (3.4), we obtain the formula

$$\beta = 1 - \eta\,\mathrm{ch}2\theta + O(\eta^2) \tag{8.7}$$

which is more accurate than (8.5). We obtain the expansion for q from Eq. (7.18). Substituting the quantity

$$\rho = \alpha + \mu = 1 + \eta\frac{\mathrm{sh}2\theta}{4\theta} - \frac{\eta}{2} + O(\eta^2)$$

obtained from conditions (8.1) and (8.2), into Eq. (7.18), and also the quantity

$$\kappa = -i\eta\frac{\mathrm{sh}2\theta}{4\theta} + O(\eta^2)$$

obtained from condition (8.2) and Eq. (7.24), and retaining terms of the first order in η, we obtain a refinement of formula (8.3)

$$q = 1 + \eta\frac{\mathrm{sh}2\theta}{2\theta} + O(\eta^2) \tag{8.8}$$

Now, substituting expansions (8.7) and (8.8) into (8.6), we obtain

$$\delta^2 = 1 + \eta\left(\frac{\mathrm{sh}2\theta}{2\theta} - \mathrm{ch}2\theta\right) + O(\eta^2)$$

Thus, collecting all the formulae, we have, as $k \to 0$,

$$f(\chi) \approx \chi\left(1 + \eta\frac{\mathrm{sh}2\theta}{4\theta}\right) - \frac{\eta}{4\theta}\sin 2\theta\chi, \quad \alpha \approx 1 + \eta\frac{\mathrm{sh}2\theta}{4\theta}$$

$$\delta \approx 1 + \frac{\eta}{4\theta}(\mathrm{sh}2\theta - 2\theta\,\mathrm{ch}2\theta), \quad F \approx \left(\frac{\mathrm{th}2\theta}{2\theta}\right)^{1/2}$$

If we make the replacement $\eta = -4\theta\epsilon$ here, we obtain formulae that are completely identical with the Stokes expansion (4.10).

9. The limit solution as $k \to 1$

We will now show that, when $k \to 1$, solution (7.8) gives solitary waves. In this case sn$u$ becomes th$u = i$tgθ, cn$u$ and dn$u$ become $1/\mathrm{ch}u = 1/\cos\theta$; $\mathbf{E} \to 0$, $\mathbf{K} \to \infty$, and hence from Eqs. (7.18)–(7.22), together with relations (7.15) and (7.24) we obtain the system

$$\left(\rho - \eta\,\mathrm{tg}^2\theta\right)^2 = q\left(1 - i\gamma\kappa - \gamma^2\kappa^2\right)$$

$$\rho\left(3\mathrm{tg}^2\theta + 1\right) - \eta\,\mathrm{tg}^2\theta\left(\mathrm{tg}^2\theta - 1\right) = -iq\left(i\gamma + 2\gamma^2\kappa\right)\frac{\mathrm{tg}\theta}{\theta}$$

$$-4\rho\,\mathrm{tg}^2\theta + \eta(\mathrm{tg}^2\theta - 1)^2 = 4q\gamma^2\eta\frac{\mathrm{tg}^2\theta}{\theta^2}$$

$$\mu + \eta\left(1 - \frac{\mathrm{tg}\theta}{\theta}\right) = 1 - \alpha, \quad \eta + \mu = 0$$

$$\rho = \alpha + \mu, \quad \kappa = 2i(1 - \alpha)$$

The solution of the system of seven equations in the seven unknowns $\mu, \eta, \rho, \kappa, q, \gamma, \alpha$ is expressed in terms of the function

$$\Delta = 1 + 2\cos 4\theta + \frac{2\sin 2\theta - \sin 4\theta}{2\theta}$$

using the formulae

$$\mu=-\eta=\frac{1-\cos 4\theta}{2\Delta},\quad \rho=\frac{3}{2}\frac{1+\cos 4\theta}{\Delta}$$

$$\kappa=\frac{8i\cos\theta\sin^3\theta}{\theta\Delta},\quad q=\frac{3+4\cos 4\theta+2\cos 8\theta}{\Delta^2}$$

$$\gamma=\frac{4\theta(2\cos 2\theta+\cos 6\theta)-\sin 2\theta+2\sin 4\theta-\sin 6\theta}{2\sin 2\theta(1+2\cos 4\theta)},\quad \alpha=\frac{1+2\cos 4\theta}{\Delta}$$

We further find the Froude number from the formula

$$F=\sqrt{1/\gamma-2(1-\alpha)}$$

obtained from relations (3.4) and (3.5). We find the quantity δ from formula (8.6), by first finding β from formula (3.4). As a result we have a solution describing solitary waves

$$f(\chi)=\frac{1+2\cos 4\theta}{\Delta}\left(\chi+\frac{1}{\theta(3\operatorname{ctg}^2 2\theta-1)}\operatorname{th}\theta\chi\right),\quad \alpha=\frac{1+2\cos 4\theta}{\Delta}$$

$$\delta=\frac{(1+2\cos 4\theta)^{3/2}}{(3-4\cos 2\theta+4\cos 4\theta)^{1/2}\Delta},\quad F=\left(\frac{\sin 2\theta-2\sin 4\theta+2\sin 6\theta}{2\theta\cos 2\theta\,\Delta}\right)^{1/2} \tag{9.1}$$

In the theory of long waves θ is a small parameter. We expand expression (9.1) in a power series in θ

$$f(\chi)=\chi\left(1-\frac{4}{3}\theta^2-4\theta^4+\ldots\right)+\left(\frac{4}{3}\theta+\frac{32}{9}\theta^3+\ldots\right)\operatorname{th}\theta\chi$$

$$\alpha=1-\frac{4}{3}\theta^2-4\theta^4+\ldots,\quad \delta=1+\frac{8}{9}\theta^4+\ldots,\quad F=1-\frac{4}{3}\theta^2-\frac{302}{45}\theta^4+\ldots$$

Comparing this with solution (6.4), obtained from long-wave theory, we see that they are completely identical.

However, solution (9.1) is more accurate than (6.4). We will show this. Suppose the liquid in a solitary wave at infinity moves horizontally in a layer of thickness h_∞ with a velocity u_∞. From formulae (9.1) and (6.4) we obtain the Froude number $F_\infty=u_\infty/\sqrt{gh_\infty}$ for infinite distances from the crest of the solitary wave $\varphi\to\pm\infty$ and we compare the values obtained with the exact value, given by the well-known Stokes formula

$$F_\infty=\sqrt{\frac{\operatorname{tg}2\theta}{2\theta}} \tag{9.2}$$

Formula (9.2) establishes a relation between the Froude number and the parameter θ, characterizing the rate at which the free surface at infinity approaches a horizontal line.

Using the formula of long-wave theory (6.3), we obtain

$$F_\infty\approx 1+\frac{2}{3}\theta^2$$

(the coefficient of θ^4 is not defined within the framework of approximation (6.3)), which, apart from terms of the second order of infinitesimals, is identical with the expansion

$$\sqrt{\frac{\operatorname{tg}2\theta}{2\theta}}=1+\frac{2}{3}\theta^2+\frac{38}{45}\theta^4+\frac{220}{1891}\theta^6+\ldots \tag{9.3}$$

However, if we use formula (9.1), it follows from them that the Froude number at infinity is related to the Froude number at the crest of the wave by the relation

$$F_\infty=F\frac{\rho-\eta\operatorname{tg}^2\theta}{(\rho+\eta)^{3/2}}$$

Hence,

$$F_\infty=(3-2\cos 2\theta+2\cos 4\theta)\sqrt{\frac{\sin 2\theta-2\sin 4\theta+2\sin 6\theta}{2\theta\cos 2\theta(1+2\cos 4\theta)^3}}$$

$$=1+\frac{2}{3}\theta^2+\frac{38}{45}\theta^4-\frac{524}{63}\theta^6+\ldots$$

We therefore have agreement with expansion (9.3) down to the fourth order of infinitesimals.

Remark. To write the initial problem on waves in dimensionless form we used the dimensional quantities h_0 and u_0 – the depth of the liquid and the velocity at the crest of the wave. If we assume that h_0 and u_0 are the depth and velocity at the trough, then, in formulating Problem 2, only the normalization condition (3.6) is changed. Solution (7.8) is not changed. The expressions for the constants α, δ and F are changed only slightly. However, if, in the new solution, the limit solitary waves is considered, Stokes relation (9.2) will be satisfied exactly.

As a whole, solution (9.1) describes the flow far from the crest of the solitary wave better than (6.3). Correspondingly, solution (7.8) more accurately describes the behaviour of the waves at a wave trough, than the formulae for cnoidal waves (5.27).

10. Conclusion

We have obtained an exact solution of non-linear boundary-value problem 4, describing steady gravitational waves of small amplitude, suitable for all wavelengths. The solution itself has turned out to be simple – it has the form (7.8). The system of five algebraic equations, relating the five constants, occurring in the solution, is more complex. It has been shown that, in the limit of long waves, solution (7.8) gives formulae of the theory of cnoidal waves, while for short waves it gives the Stokes expansion.

The possibility of extending the solution obtained is important. In boundary condition (7.3) we retained the squares of the amplitudes and dropped terms of higher order of infinitesimals. However, we can retain all terms up to the N-th order of infinitesimals in amplitude, i.e., replace boundary condition (7.3) in Problem 4 by

$$\left|\frac{dW}{d\chi}+\alpha\right|^2 = q\sum_{j=0}^{N}(2\gamma \operatorname{Im} W)^j$$

The new boundary-value thereby obtained will describe waves of high amplitude better. Its solution can be obtained by a method similar to that used to solve Problem 4, i.e., by reduction to a system of ordinary differential equations.

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